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THEORETICAL ARTICLE

Effect of memory on an inventory model for deteriorating item: fractional calculus approach

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Abstract

It is incontrovertible that involvement of memory has a great impact in inventory model. For any company, long past experience as well as short past experience have similar significant importance to manage the profit. Our proposed inventory model is leaded by two important factors: constant demand and deterioration. Here, we have introduced memory effect through the feasible ideas of fractional calculus. Also, we consider the order of fractional derivative as memory index. We calculate various type of costs viz total holding cost, purchasing cost, deterioration cost, shortage cost, salvage value. Additionally, optimal ordering interval, optimal starting shortage time and minimized total average cost are computed theoretically using the fractional calculus techniques. Effect of memory is justified by choosing a suitable numerical example. Finally sensitivity analysis for the model has been presented.

Keywords Fractional differential equation · Fractional laplace transform · Mittg-Leffler function · Memory · Salvage value

Mathematics Subject Classification 90B50 · 90C31 · 34A08 · 26A33

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1 Introduction

Fractional Calculus (FC), generalization of traditional classical differentiation, integration and differential equation [17], is a most emerging tool for the recent research in the field of mathematics. Due to the insufficiency of logical interpretation of fractional-order differential equations, the FC had been employed extremely infrequently until a few days ago. Now a days, FC is successfully used to address challenging issues in science and technology due to its enormous knowledge expansion. Although, the formulation of basic ideas and their physical interpretations are constantly being modified. FC can interpret physical occurrence in its generalized form quite effectively. According to [8, 14, 17], Fractional Differential Equation (FDE) has some of the most useful features and aspects for solving many complex problems found in everything viz applied sciences and real-world problems. Keeping past effects of past events is the most significant physical acceptability of FC. In order to describe physical problems naturally and realistically, the effects of past events cannot be overlooked while defining the current situation. In biology and economics, FC has been utilized as a perfect tool for understanding mathematically the effect of memory measured by memory index defined by the fractional order derivatives [28–32]. It has been established as the most powerful mathematical method to understand our true nature. Thereby, numerous researchers are working in the attraction of FC for last three decades.

The literature review on inventory management began decades ago. The study of inventory management has recently emerged as one of the primary subfields of operation research. Demand pattern is important in the inventory model and is thoroughly researched by various researchers. Jana and Das [13] have discussed an inventory model where the demand depends on stock and deterioration occurred. Mishra and Saha [18] reported the inventory model with time dependent demand. Dye et al. [9] worked on the inventory model where the demand is time-varying and partial backlogging is shortage-dependent. Also, [11, 12] considered time varying demand in his inventory models. There is no doubt that deterioration [6] has a substantial impact on the whole inventory system. Deterioration happens in various ways, such as melting, damage, decay, etc. The inventory systems are developed in the consideration of constant deterioration with the demand, dependent on time [10, 16], dependent on stock [13, 33], dependent on price [26], etc. Singh and Pattnayak [27] introduced the variable deterioration in inventory model. Researchers (like [16, 22]) have examined the effects of item shortage. Partial backlogging is another important component in an inventory system and is reported by [1, 2], and [27] with a variety of intriguing outcomes.

Demand from buyers is influenced by seller behaviour, delivery time, product quality, and pricing. The purchase of a product by a customer relies upon the product's past history. The customers do not intend to buy the product with poor feedbacks. So, in a business policy, memory i.e. past experience has a tremendous impact and must be included in the development of strategic quality business policy. In contrast to our standard differential equation of integer order, the most significant benefit of FDE is that it can be used as a perfect tool to more accurately characterize the dynamical behavior of any system with the implicit incorporation of the idea of memory. Das and Roy [3–5], Das et al. [7] have reported the potential use of FC in a variety of inventory models and established several hypotheses. Pakhira et al. [19–21] attempted to apply the memory idea to an inventory system with the help of FC. Jana and Das [13] imposed memory in an inventory model with stock dependent demand. They have reported that their modified inventory model can be treated as an effective and efficient real world realistic optimization problem. Very recently, Rahaman et al. [25] have studied an inventory management problem where they have reported that how a retailer's decision is influenced by the memory. In another recent paper, Kumar et al. [15] proposed the influential role of memory effect of promotional efforts in an inventory system through the study of FC with time-dependent demand. Thereby, we are motivating to study the effect of memory on an inventory model for deteriorating item with fractional calculus approach. Shortage is natural phenomena in an inventory system and it plays a significant role in the optimization of profit. It is not yet been investigated in the presence of memory effect how starting shortage time effects on the minimized total average cost in an inventory system. It has inspired us to look into the starting shortage time in an inventory system.

Our goal is to improve the performance of our current inventory models by including the effect of memory or prior experience. Here the memory effect is considered on an inventory model with constant demand and deterioration. Shortages occurs after some time period. Our objective is that to find out the optimum ordering interval and optimum starting shortage time for which the total average cost is minimum when an inventory system is subjected to memory effects via a differential equation of fractional order. To incorporate memory effect, we use FC. Here we consider fractional order derivative as memory index. Fractional order derivative, integration and FDE, provided by Podlubny [23] and Miller and Ross [17], are used to calculate theoretically total holding cost, purchasing cost, deterioration cost, shortage cost, salvage value, optimal ordering interval, optimal starting shortage time and minimized total average cost. Mainly we want to explore in the presence of memory effect how the starting shortage time effects on minimized total average cost.

The article then goes on as follows:

Some mathematical preliminaries and definitions is presented in Sect. 2. Long memory effect and short memory effect are discussed here. Section 3 presents the creation of the classical inventory model with "notation" and "assumption". The generalized inventory model for deteriorating item with fractional order is introduced herein Sect. 4. The solutions and analysis of the formulated inventory model with fractional order are presented in the preceding. Also in this section various relevant costs are calculated. Section 5 discusses the numerical illustration of the suggested model, and Sect. 6 is concluded with a conclusion.

2 Some mathematical preliminaries and definitions

Fractional derivative has numerous definitions. They each have their own unique physical interpretation [24]. We will introduce the Riemann–Liouville definition and the Caputo definition of fractional derivative in this section. Moreover, the fractional Laplace transform method is also introduced to develop the article. These are briefly described in the following.

2.1 Riemann–Liouville (R–L) fractional derivative

The definition of Riemann–Liouville (R–L) fractional derivative of order α (where $m \le \alpha < m+1$) for a integrable function $f(x)$ defined on $[a, b]$ is denoted by ${}_{a}^{RL}D_{x}^{\alpha}(f(x))$ and is defined by

$$
{}_{a}^{RL}D_{x}^{\alpha}(f(x)) = \frac{1}{\Gamma(m+1-\alpha)}\left(\frac{d}{dx}\right)^{m+1}\int_{a}^{x}(x-\xi)^{m-\alpha}f(\xi)d\xi.
$$

The above definition is known as the left R–L αth order fractional derivative and the definition of right R–L αth order fractional derivative is as follow,

$$
{}_{x}^{RL}D_{b}^{\alpha}(f(x)) = \frac{1}{\Gamma(m+1-\alpha)}\left(-\frac{d}{dx}\right)^{m+1}\int_{x}^{b}(x-\xi)^{m-\alpha}f(\xi)d\xi.
$$

The aforementioned finding establishes a distinction between fractional derivatives and ordinary derivatives. For any differentiable functions, M Caputo offered a new concept of fractional order derivative to eliminate this discrepancy.

2.2 Caputo type fractional order derivative

The definition of Caputo fractional derivative of order α (where $m \le \alpha < m+1$) for a integrable function $f(x)$ defined on $[a, b]$ is denoted by ${}_{a}^{C}D_{x}^{\alpha}(f(x))$ and is defined by

$$
{}_{a}^{C}D_{x}^{\alpha}(f(x)) = \frac{1}{\Gamma(m+1-\alpha)}\int_{a}^{x}(x-\xi)^{m-\alpha}f^{(m+1)}(\xi)d\xi.
$$

Only differentiable functions are suitable to the Caputo type fractional derivative. However, the fractional derivative for a constant function is zero according to Caputo's definition. Additionally, the initial as well as boundary conditions must be identical to those of a classical i.e, integer order differential equation in order to solve Caputo type fractional order differential equations.

2.3 Fractional Laplace Transform (FLT) method

In differential equations of integer and fractional order, the Laplace Transform(LT) is crucial. $F(s)$ (where $s > 0$) denotes the LT for the function $f(t)$ and $F(s)$ is defined by

$$F(s) = L(f(t)) = \int_0^{\infty} e^{-st} f(t)dt.$$

Also, $L(f^r(t))$ denotes the LT for the the function $f^r(t)$ and is defined by

$$L(f^r(t)) = s^r F(s) - \sum_{p=0}^{r-1} s^{r-p-1} f^p(0),$$

where the r-th order integer derivative of $f(t)$ is $f^r(t)$. For the αth order fractional derivative of $f(t)$, the FLT is denoted by $L(f^\alpha(t))$ and is defined by

$$L(f^\alpha(t)) = s^\alpha F(s) - \sum_{p=0}^{r-1} s^p f^{\alpha-p-1}(0),$$

where $r - 1 < \alpha \leq r$.

2.4 Mittag-Leffler function

In the theory of differential equations of integer order, the exponential function, e^z, is crucial. Its single parameter generalization, the function is indicated by $E_\alpha(z)$ and is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k + 1)}.$$

Also, a two-parameter Mittag-Leffler type function is defined as follows,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k + \beta)}.$$

2.5 Long memory effect

The memory strength is controlled by fractional order derivative. If the order of the fractional derivative lies in (0, 0.5). Then the system is called of long memory effected.

2.6 Short memory effect

If the order of fractional derivative lies in [0.5, 1). Then the system is called of short memory effected.

3 Classical inventory model

On the basis of the assumptions given below, we prepare the classical inventory model (Fig. 1).

3.1 Notations

To develop the proposed generalized model, the following notations are being made:

(i) a: Constant demand rate per unit time during the cycle.
(ii) q(t): Inventory level at time t.
(iii) C_1: Constant inventory holding cost per unit quantity per unit time.
(iv) C_2: Constant purchasing cost of per unit item per unit time.
(v) C_3: Constant deteriorating cost of per unit item per unit time.
(vi) C_4: Per unit shortage cost per unit item per unit time.
(vii) C_γ: Salvage value parameter per unit, associated with deteriorated units during the cycle time.
(viii) θ: Constant deterioration rate, units/unit time, where $0 < \theta << 1$.
(ix) A: Constant ordering cost
(x) t_1: Time at which shortage start ($t_1 > 0$).
(xi) T: Length of each ordering cycle.
(xii) W: Initial inventory level for each ordering cycle.
(xiii) S: Maximum amount of demand backlogged for each ordering cycle.
(xiv) Q: Economic order quantity for each ordering cycle.
(xv) $HC_\alpha(T, t_1)$: Total Holding cost per cycle.

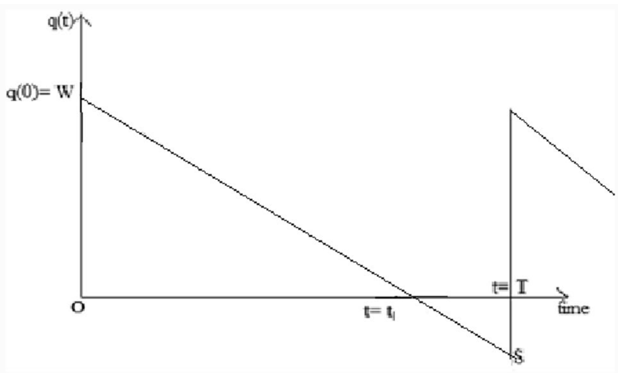

Fig. 1 Graphical representation of the inventory model

(xvi) $PC_\alpha(T, t_1)$: Total Purchasing cost per cycle.
(xvii) $DC_\alpha(T, t_1)$: Total Deteriorating cost per cycle.
(xviii) $SC_\alpha(T, t_1)$: Total Shortage cost per cycle.
(xix) $SV_\alpha(T, t_1)$: Total Salvage value of deteriorated item.
(xx) $TC_\alpha(T, t_1)$: Total Inventory cost per cycle.
(xxi) $TAC_\alpha(T, t_1)$: Total average Inventory cost per unit time per cycle.
(xxii) T^*: Optimal value of T.
(xxiii) TAC_α^*: Minimized total average cost.
(xxiv) $\Gamma(.)$: Gamma Function.
(xxv) $E_\alpha(z)$: Mittag-Leffler Function with one parameter.

3.2 Assumptions

To develop the proposed generalized model, the assumptions which are being made is given below:

1. A single item is considered over the fixed period T which is subject to a constant deterioration rate.
2. There is no replenishment or repair of deteriorated items takes place in a given cycle.
3. The consumption rate or demand rate at any time t is constant.
4. The planning horizon is infinite. Only a typical planning schedule of length is considered and all the remaining cycles are identical.
5. The replenishment occurs instantaneously at an infinite rate and lead time is zero.
6. The holding cost per unit, purchasing cost per unit, shortage cost per unit, deteriorating cost per unit, salvage value per unit item are constant.
7. The ordering cost is constant.
8. Shortages are allowed during stock out period and are completely backordered.

4 Generalized inventory model

We consider the fractional order generalized inventory model with constant demand. Replenishment occurs at time t = 0 when the inventory level attains its maximum, W. From $t = 0$ to t_1, the inventory level reduces due to demand and deterioration. At time t_1, the inventory level achieves zero, then shortage is allowed to occur during the time interval $[t_1, T]$, and all of the demand during shortage period $[t_1, T]$ partially backlogged. As the inventory level reduces due to demand rate as well as deterioration during the inventory interval $[t_1, T]$. Therefore, the behavior of this inventory system at any time t can be represented by the following picture:

Based on the above description, during the time interval [0, T] the differential equation representing the proposed inventory status is given by

$$\frac{d(q(t))}{dt} + \theta q(t) = -a \quad \text{for} \quad 0 \le t \le t_1, \tag{1}$$

$$\frac{d(q(t))}{dt} = -a \quad \text{for} \quad t_1 \le t \le T, \tag{2}$$

with $q(0) = W$, $q(t_1) = 0$, $q(T) = -S$.

Equations (1), (2) can be written in terms of Caputo fractional derivative sense as

$${}_a^C D_t^\alpha q(t) + \theta q(t) = -a \quad \text{for} \quad 0 \le t \le t_1, \tag{3}$$

$${}_a^C D_t^\alpha q(t) = -a \quad \text{for} \quad t_1 \le t \le T. \tag{4}$$

Solving the Eq. (3) with the initial condition $q(0) = W$ and using the Laplace transform method for solving fractional differential equation we get,

$$q(t) = WE_{\alpha,1}(-\theta t^\alpha) - at^\alpha E_{\alpha,\alpha+1}(-\theta t^\alpha) \qquad \text{for} \ \ 0 \le t \le t_1, \tag{5}$$

$$= W\left[1 - \frac{\theta t^\alpha}{\Gamma(\alpha+1)} + \frac{\theta^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right] - at^\alpha\left[\frac{1}{\Gamma(\alpha+1)} - \frac{\theta t^\alpha}{\Gamma(2\alpha+1)} + \frac{\theta^2 t^{2\alpha}}{\Gamma(3\alpha+1)}\right], \tag{6}$$

$$= W - (W\theta + a)\frac{t^\alpha}{\Gamma(\alpha+1)} + (W\theta^2 + a\theta)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - a\theta^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \tag{7}$$

$q(t_1) = 0$ gives

$$W = \frac{at_1^\alpha\left[\frac{1}{\Gamma(\alpha+1)} - \frac{\theta t_1^\alpha}{\Gamma(2\alpha+1)} + \frac{\theta^2 t_1^{2\alpha}}{\Gamma(3\alpha+1)}\right]}{\left[1 - \frac{\theta t_1^\alpha}{\Gamma(\alpha+1)} + \frac{\theta^2 t_1^{2\alpha}}{\Gamma(2\alpha+1)}\right]},$$

$$= at_1^\alpha\left[\frac{1}{\Gamma(\alpha+1)} - \frac{\theta t_1^\alpha}{\Gamma(2\alpha+1)} + \frac{\theta^2 t_1^{2\alpha}}{\Gamma(3\alpha+1)}\right]\left[1 - \left(\frac{\theta t_1^\alpha}{\Gamma(\alpha+1)} - \frac{\theta^2 t_1^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right]^{-1},$$

$$= at_1^\alpha\left[\frac{1}{\Gamma(\alpha+1)} - \frac{\theta t_1^\alpha}{\Gamma(2\alpha+1)} + \frac{\theta^2 t_1^{2\alpha}}{\Gamma(3\alpha+1)}\right]\left[1 + \frac{\theta t_1^\alpha}{\Gamma(\alpha+1)} - \frac{\theta^2 t_1^{2\alpha}}{\Gamma(2\alpha+1)} + \left(\frac{\theta t_1^\alpha}{\Gamma(\alpha+1)} - \frac{\theta^2 t_1^{2\alpha}}{\Gamma(2\alpha+1)}\right)^2\right],$$

$$= at_1^\alpha\left[\frac{1}{\Gamma(\alpha+1)} + \theta t_1^\alpha\left(\frac{1}{(\Gamma(\alpha+1))^2} - \frac{1}{\Gamma(2\alpha+1)}\right) + \theta^2 t_1^{2\alpha}\left(\frac{1}{(\Gamma(\alpha+1))^3} - \frac{2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)}\right)\right].$$

Again solving the Eq. (4) with the initial condition $q(t_1) = 0$ and using the Laplace transform method, we get

$$q(t) = -a\frac{(t-t_1)^\alpha}{\Gamma(\alpha+1)} \qquad \text{for} \ \ t_1 \le t \le T. \tag{8}$$

$q(T) = -S$ gives

$$S = a\frac{(T-t_1)^\alpha}{\Gamma(\alpha+1)}.$$

Therefore the economic order quantity (EOQ) is $Q = W + S$.

Thus,

$$Q = at_1^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + \theta t_1^{\alpha}\left(\frac{1}{(\Gamma(\alpha+1))^2} - \frac{1}{\Gamma(2\alpha+1)}\right) + \theta^2 t_1^{2\alpha}\left(\frac{1}{(\Gamma(\alpha+1))^3} - \frac{2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)}\right)\right] + a\frac{(T-t_1)^{\alpha}}{\Gamma(\alpha+1)}. \tag{9}$$

4.1 Various relevant costs

Holding Cost: In an inventory costs related to unsold item are known as holding costs. A company's holding expenses consist of the cost of damaged or spoilt items as well as labour, storage space, and insurance.

Generalized total Holding cost is $HC_{\alpha}(T, t_1)$ and is defined by

$$\begin{aligned} HC_{\alpha}(T,t_1) &= C_1\int_0^{t_1} q(t)dt, \\ &= C_1\int_0^{t_1}\left[W - (W\theta + a)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (W\theta^2 + a\theta)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - a\theta^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right]dt, \\ &= C_1\left[Wt_1 - (W\theta + a)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (W\theta^2 + a\theta)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - a\theta^2\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right]. \end{aligned}$$

Purchasing cost: Purchasing cost refers to the complete cost of the good or service, including any applicable taxes, shipping charges, additional fees, and contingencies.

Total Purchasing cost per cycle is $PC_{\alpha}(T, t_1)$ and is defined by

$$\begin{aligned} PC_{\alpha}(T,t_1) &= C_2W + C_2\int_{t_1}^{T} adt, \\ &= C_2W + C_2a(T - t_1). \end{aligned}$$

Deteriorating Cost: Deterioration is the inability to use an item for its intended use due to decay, damage, evaporation, spoiling, obsolescence, loss of utility, pilferage, or loss of marginal values of a commodity.

Total Deteriorating Cost per cycle is $DC_{\alpha}(T, t_1)$ and is defined by

$$\begin{aligned} DC_{\alpha}(T,t_1) &= \theta C_3\int_0^{t_1} q(t)dt, \\ &= \theta C_3\left[Wt_1 - (W\theta + a)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (W\theta^2 + a\theta)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - a\theta^2\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right]. \end{aligned}$$

Shortage Cost: In inventory costs associated with shortages are those incurred when a company runs out of stock, such as: Time lost for the unavailability of raw materials. Cost of obsolescence, theft, and shrinking.

Total Shortage Cost per cycle is $SC_\alpha(T,t_1)$ and is defined by

$$\begin{aligned} SC_\alpha(T,t_1) &= -C_4\int_{t_1}^{T} q(t)dt, \\ &= aC_4\frac{(T-t_1)^{\alpha+1}}{\Gamma(\alpha+2)}. \end{aligned}$$

Salvage value: Salvage value is an asset's estimated book value after completion of depreciation.

Total Salvage value per cycle is $SV_\alpha(T,t_1)$ and is defined by

$$\begin{aligned} SV_\alpha(T,t_1) &= \theta C_\gamma \int_0^{t_1} q(t)dt, \\ &= \theta C_\gamma \left[Wt_1 - (W\theta + a)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (W\theta^2 + a\theta)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - a\theta^2\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \right]. \end{aligned}$$

Total generalized Inventory Cost is

$$\begin{aligned} TC_\alpha(T,t_1) &= A + HC_\alpha(T,t_1) + PC_\alpha(T,t_1) + DC_\alpha(T,t_1) + SC_\alpha(T,t_1) - SV_\alpha(T,t_1), \\ &= A + a(C_1 + \theta C_3 - \theta C_\gamma)\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right)t_1^{\alpha+2} \\ &\quad + a\theta(C_1 + \theta C_3 - \theta C_\gamma)\left(\frac{1}{\Gamma(2\alpha+2)} + \frac{1}{(\Gamma(\alpha+1))^2} - \frac{1}{\Gamma(2\alpha+1)} \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+2)}\right)t_1^{2\alpha+1} \\ &\quad + a\theta^2(C_1 + \theta C_3 - \theta C_\gamma)\left(\frac{1}{(\Gamma(\alpha+1))^3} - \frac{2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)} \right. \\ &\quad - \frac{1}{(\Gamma(\alpha+1))^2\Gamma(\alpha+2)} \\ &\quad \left. + \frac{1}{\Gamma(\alpha+2)\Gamma(2\alpha+1)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2\alpha+2)} - \frac{1}{\Gamma(3\alpha+2)}\right)t_1^{3\alpha+1} \\ &\quad + C_2a(T-t_1) + C_4a\frac{(T-t_1)^{\alpha+1}}{\Gamma(\alpha+2)} \\ &\quad + C_2at_1^\alpha\left[\frac{1}{\Gamma(\alpha+1)} + \theta t_1^\alpha\left(\frac{1}{(\Gamma(\alpha+1))^2} - \frac{1}{\Gamma(2\alpha+1)}\right) - \frac{\theta^2 t_1^{2\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]. \end{aligned}$$

Therefore the generalized Average Inventory Cost per unit time is denoted by $TAC_\alpha(T,t_1)$ and is defined by

$$
\begin{aligned}
TAC_\alpha(T,t_1) =& \frac{1}{T}[TC_\alpha(T,t_1)]\\
=& \Big[A + a(C_1+\theta C_3-\theta C_\gamma)\left(\frac{1}{\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+2)}\right)t_1^{\alpha+2}\\
&+ a\theta(C_1+\theta C_3-\theta C_\gamma)\left(\frac{1}{\Gamma(2\alpha+2)} + \frac{1}{(\Gamma(\alpha+1))^2} - \frac{1}{\Gamma(2\alpha+1)}\right.\\
&\left. - \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+2)}\right)t_1^{2\alpha+1} + a\theta^2(C_1+\theta C_3-\theta C_\gamma)\left(\frac{1}{(\Gamma(\alpha+1))^3} - \frac{2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right.\\
&+ \frac{1}{\Gamma(3\alpha+1)} - \frac{1}{(\Gamma(\alpha+1)\Big)^2\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+2)\Gamma(2\alpha+1)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}\\
&\left. - \frac{1}{\Gamma(3\alpha+2)}\right)t_1^{3\alpha+1} + C_2a(T-t_1) + C_4a\frac{(T-t_1)^{\alpha+1}}{\Gamma(\alpha+2)} + C_2at_1^\alpha\Big[\frac{1}{\Gamma(\alpha+1)}\\
&+ \theta t_1^\alpha\left(\frac{1}{(\Gamma(\alpha+1))^2} - \frac{1}{\Gamma(2\alpha+1)}\right) - \frac{\theta^2 t_1^{2\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\Big]\Big]T^{-1}.
\end{aligned}
$$

Therefore, the memory dependent proposed fractional order inventory model written as following

$$
\begin{cases}
\text{Min} & TAC_\alpha(T,t_1)\\
& = \Big[A + a(C_1+\theta C_3-\theta C_\gamma)\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right)t_1^{\alpha+2}\\
& \quad + a\theta(C_1+\theta C_3-\theta C_\gamma)\left(\frac{1}{\Gamma(2\alpha+2)} + \frac{1}{(\Gamma(\alpha+1))^2} - \frac{1}{\Gamma(2\alpha+1)}\right.\\
& \quad \left. - \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+2)}\right)t_1^{2\alpha+1} + a\theta^2(C_1+\theta C_3-\theta C_\gamma)\left(\frac{1}{(\Gamma(\alpha+1))^3} - \frac{2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right.\\
& \quad + \frac{1}{\Gamma(3\alpha+1)} - \frac{1}{(\Gamma(\alpha+1))^2\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+2)\Gamma(2\alpha+1)} + \frac{1}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}\\
& \quad \left. - \frac{1}{\Gamma(3\alpha+2)}\right)t_1^{3\alpha+1} + C_2a(T-t_1) + C_4a\frac{(T-t_1)^{\alpha+1}}{\Gamma(\alpha+2)} + C_2at_1^\alpha\Big[\frac{1}{\Gamma(\alpha+1)}\\
& \quad + \theta t_1^\alpha\left(\frac{1}{(\Gamma(\alpha+1))^2} - \frac{1}{\Gamma(2\alpha+1)}\right) - \frac{\theta^2 t_1^{2\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\Big]\Big]T^{-1}\\
\text{Subject to} & T \ge 0,\\
& t_1 \ge 0.
\end{cases}
\tag{10}
$$

Table 1 Table for optimal value of starting shortage time, t_1^*, optimal ordering interval, T_α^*, and minimized total average cost, TC_α^*, different values of memory index α $(0.4 \le \alpha \le 1)$

α	t_1^*	T_α^*	TC_α^*
0.4	7.4094	68.2842	21.5938
0.5	3.4939	36.7903	23.9188
0.6	1.2153	18.6357	26.2099
0.7	0.6440	13.6575	29.0938
0.8	0.5107	10.6984	31.0463
0.9	0.4753	8.5801	32.9927
1.0	0.4550	7.4001	34.4225

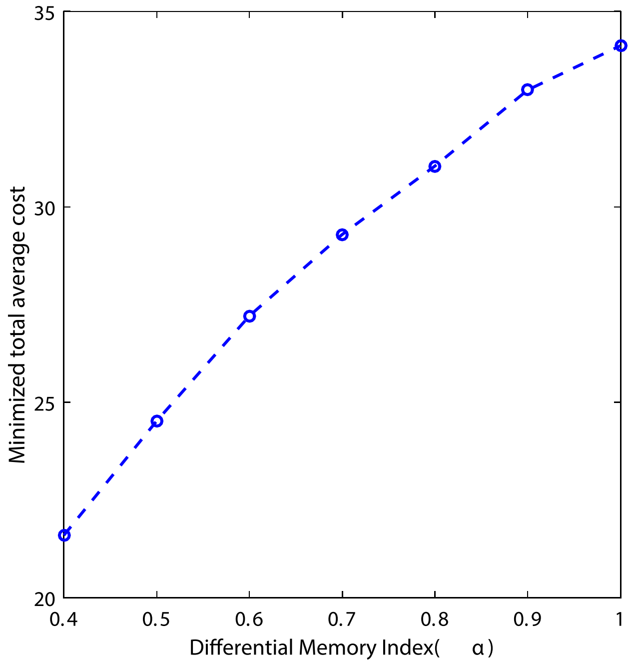

Fig. 2 Graph of memory index α and minimized total average cost

In order to estimate various costs, starting shortage time, ordering interval for different values of memory parameters assigned in the proposed model, a numerical example with some experimental data of the parameters is now shown.

5 Numerical examples

Here we illustrate the inventory model considering a numerical example. The values of parameters with proper units given below:

$$A = 60; a = 12; \theta = 0.05;$$
$$C_1 = 0.5; C_2 = 1.5; C_3 = 3; C_4 = 2.5; C_\gamma = 2.8$$

Then using Matlab Programming we get the optimum value of starting shortage time t_1^*, optimum total cycle time T_α^* and minimized total average cost TC_α^* of the considered memory dependent inventory model and the outcomes are given in Table 1.

We notice from Table 1, and Fig. 2 total average cost achieves its minimum for $\alpha = 0.4$ and attains its maximum for $\alpha = 1$ i.e memory dependent inventory model has minimum total average cost in comparison to the independent of memory. The minimized total average cost becomes negative for $\alpha < 0.4$. This means that there is no effective business for $\alpha < 0.4$. So, we count memory effect from $\alpha = 0.4$. The

Table 2 Table for sensitivity analysis of the parameters for $\alpha = 0.4$

Parameter	Parameter change (%)	t_1^*	T_α^*	TAC_α^*
A	+50	7.3882	68.0793	22.0331
	+10	7.3797	67.9428	21.6816
	−10	7.4390	68.6256	21.5059
	−50	7.4316	68.4891	21.1544
a	+50	4.2974	40.1626	31.0266
	+10	6.4502	62.4812	23.4803
	−10	7.6485	74.0872	19.7072
	−50	10.5213	97.5213	12.1610
θ	+50	7.1903	69.7853	21.1327
	+10	7.0776	68.6257	21.5089
	−10	7.0212	67.9428	21.6749
	−50	6.9084	66.7720	21.9628
C_1	+50	7.0705	68.4893	21.5203
	+10	7.0536	68.3320	21.5791
	−10	7.0451	68.2364	21.6085
	−50	7.0282	68.0795	21.6672
C_2	+50	4.3706	40.9706	29.9543
	+10	6.4854	62.1387	23.2659
	−10	7.6133	74.4321	19.9217
	−50	9.7281	95.5979	13.2332
C_3	+50	7.0564	68.3525	21.5717
	+10	7.0508	68.2979	21.5894
	−10	7.0480	68.2706	21.5982
	−50	7.0423	68.2160	21.6158
C_4	+50	6.3444	61.2511	23.6863
	+10	6.9154	66.9868	22.0123
	−10	7.1833	69.5816	21.1753
	−50	7.7543	75.3212	19.5013
C_γ	+50	5.4280	53.2616	26.5379
	+10	6.7674	65.5529	22.4523
	−10	7.3314	71.0157	20.7204
	−50	8.6707	83.3067	16.8745

system-imposed memory steadily improves the optimal ordering interval, which achieves its maximum value at its robust memory level. The company policy will therefore be stable and continue to be extensive for deep memory level, which is extremely thorough. This might occur as a result of their previous experience's broad prudence. As a result, the memory dependent generalized inventory model has a broad vision of advantage that will be taken into account right away in light of a consistent company policy. Additionally, long memory affected systems have

Table 3 Table for sensitivity analysis of the parameters for $\alpha = 0.9$

Parameter	Parameter change (%)	t_1^*	T_α^*	TAC_α^*
A	+50	0.4249	7.6706	36.4892
	+10	0.4658	8.4085	33.6920
	−10	0.4848	8.7517	32.2934
	−50	0.5257	9.4860	29.4963
a	+50	0.3422	6.2206	42.3309
	+10	0.4482	8.0825	34.8604
	−10	0.5024	9.0777	31.1251
	−50	0.6084	10.9396	23.6545
θ	+50	0.4749	8.5724	33.0223
	+10	0.4752	8.5784	32.9985
	−10	0.4754	8.5818	32.9870
	−50	0.4757	8.5878	32.9649
C_1	+50	0.4748	8.5715	33.0405
	+10	0.4751	8.5775	33.0023
	−10	0.4754	8.5827	32.9832
	−50	0.4758	8.5887	32.9649
C_2	+50	0.3470	6.2635	42.0593
	+10	0.4487	8.1082	34.8060
	−10	0.5019	9.0520	31.1794
	−50	0.6036	10.8967	23.9262
C_3	+50	0.4751	8.5767	33.0071
	+10	0.4752	8.5801	32.9956
	−10	0.4754	8.5802	32.9899
	−50	0.4755	8.5835	32.9784
C_4	+50	0.4183	7.5505	36.8640
	+10	0.4639	8.3742	33.7670
	−10	0.4867	8.7860	32.2185
	−50	0.5323	9.6097	29.1215
C_γ	+50	0.4040	7.2931	37.9416
	+10	0.4582	8.2712	34.1803
	−10	0.4924	8.8889	31.8051
	−50	0.5466	9.8671	28.0438

a wider optimal ordering interval than memory less systems and have too little total average cost that is minimized when compared to memory less systems.

5.1 Sensitivity analysis

A sensitivity analysis has been carried out to demonstrate the implications of change in system parameters A, a, θ, C_1, C_2, C_3, C_4, C_γ on the optimal ordering interval

Table 4 Table for the impact of changing parameters on the minimized total average cost at $\alpha = 0.4$

Parameter	Parameter change (%)	Change of minimized total average cost (%)
A	−50 to −10	1.662
	−10 to +10	0.817
	+10 to +50	1.621
a	−50 to −10	62.052
	−10 to +10	19.146
	+10 to +50	32.139
θ	−50 to −10	1.310
	−10 to +10	0.766
	+10 to +50	1.749
C_1	−50 to −10	0.270
	−10 to +10	0.136
	+10 to +50	0.272
C_2	−50 to −10	50.543
	−10 to +10	16.787
	+10 to +50	28.748
C_3	−50 to −10	0.081
	−10 to +10	0.041
	+10 to +50	0.082
C_4	−50 to −10	8.584
	−10 to +10	3.953
	+10 to +50	7.605
C_γ	−50 to −10	22.791
	−10 to +10	8.358
	+10 to +50	18.197

T_α^*, optimal starting shortage time t_α^*, and minimized total average cost TAC_α^*. The parameters are increased by 10%, 50% also decreased by 10%, 50%, separating each parameter and leaving the others constant for various memory indexes $\alpha = 0.4$ and $\alpha = 0.9$.

From Tables 2, 3, 4 and 5, the conclusions about sensitivity analysis has been derived as follows:

(i) For the proposed inventory model the parameters a, C_2 are most sensitive.
(ii) Also the parameters a, C_2, C_γ for weak as well as strong memory are most sensitive.
(iii) For the proposed inventory model with memory the parameters θ, C_1, C_3 for weak as well as strong memory are not sensitive
(iv) The inventory model heavily relies on the constant demand a. A drop in a causes a decrease in the total minimized cost, whereas a rise in a causes an increase in the minimized total average cost.

Table 5 Table for the impact of changing parameters on the minimized total average cost at $\alpha = 0.9$

Parameter	Parameter change (%)	Change of minimized total average cost (%)
A	−50 to −10	9.483
	−10 to +10	4.331
	+10 to +50	8.302
a	−50 to −10	31.582
	−10 to +10	12.001
	+10 to +50	21.430
θ	−50 to −10	0.067
	−10 to +10	0.035
	+10 to +50	0.072
C_1	−50 to −10	0.116
	−10 to +10	0.058
	+10 to +50	0.116
C_2	−50 to −10	30.315
	−10 to +10	11.631
	+10 to +50	20.839
C_3	−50 to −10	0.035
	−10 to +10	0.017
	+10 to +50	0.035
C_4	−50 to −10	10.635
	−10 to +10	4.806
	+10 to +50	9.172
C_γ	−50 to −10	13.412
	−10 to +10	7.468
	+10 to +50	11.004

(v) For the proposed inventory model the parameters A, C_4 are mild sensitive.

6 Conclusions

The relevance of memory effects in inventory related problem has been studied here. It is noted that the foremost objective of any inventory related problem is to maximize the profit of some business policy by minimizing the entire total cost. It is observed from our study that memory effect (past experience) has fundamental role in inventory related problem and which helps policy maker to chose appropriate decision to achieve the sustainable business policy. There is no doubt that an inventory related problem with total average cost, the optimal ordering interval and starting shortage time gives better outcome for the consideration of memory effect. We have incorporated the concept of memory in the concerned inventory model with the help of fractional calculus. The fractional order of the differentiation is the indicator of the memory index in the whole model. It is observed that the memory affects effectively on total average cost and long memory effect will give powerful result in comparison of short memory or memoryless system. Here we also calculated the starting shortage time and ordering interval for which the total average cost is minimum. A supporting numerical example is provided here to demonstrate our notion from a practical standpoint. The results show that the memory dependent inventory model has more significant results than our traditional classical inventory model.

Here, the input data have been taken into account to hypothetically demonstrate the significance of the suggested model. The findings are easily applicable to real-world situations, but further experimental research is needed to show how to approach this issue effectively. Therefore, it cannot always be guaranteed that the suggested model will provide a result that is practical for any given set of presented data during any experimental research. Instead, it can be guaranteed that using the data with this kind of suggested numerical value, the current model will undoubtedly produce the best results. Practically speaking, a number of problems that arise in actual life situations could have an impact on how radical the presented data is. In the current study, it is primarily recommended to investigate optimal total costs, optimal ordering intervals, optimal starting shortage time in the presence of various memory indices. A decision-maker can gain some knowledge or ideas about the properties of parameters from the entire debate and results.

Basically memory-dependent inventory models can use the historical data to optimize order quantities and reorder points. By analyzing past demand patterns, these models can better predict future demand and adjust inventory level accordingly. The suggested inventory model gives an inventory manager the right decision-support for choosing the most cost-effective quantities of replenishment orders for deteriorating items while taking customer purchasing behavior into account. Results from our study can be used to manage inventories in situations that are comparable. For instance, decisions on inventory replenishment for circumstances where a production company produces a perishable good for sale in its own retail outlets can also be addressed. Any on-line retailers and e-marketing companies promoting a product will inevitably take the customer memory effect into serious consideration; in this regard, present model will be quite useful. When a product is not in stock in an on-line store, the consumer places an order, and the product is subsequently stocked and shipped to the client and also for the items where expiration dates are critical, deterioration costs, holding costs, and waste can all be reduced. Comparing the two inventory models from a practical standpoint, the fractional order inventory model is more helpful in the actual system.

Future study will take into account the wide range of works related to the memory dependent inventory model in numerous aspects. More research is necessary to examine the properties of the suggested memory-dependent model while imposing various practical factors, such as reliability, promotional effort, trade credit policy, and others. In various uncertain environments (such as fuzzy, neutrosophic, etc.), other approaches to this memory-dependent inventory model can be devised to express the uncertainty for the purpose of a more realistic sense.

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